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广义 Chaplygin 气体二相流方程组的黎曼问题

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摘要: 本文细致地研究了状态方程为广义 Chaplygin 气体的等熵二相流漂移通量方程,它的黎曼解由一个狄拉克激波或激波、稀疏波及接触间断的不同组合组成.在狄拉克激波解中,狄拉克 δ-函数同时叠加在两个密度之上.两个密度的狄拉克 δ-函数的强度和狄拉克激波的传播速度可以由广义推导出的 Rankine-Hugoniot 条件来计算.

关键词: 黎曼问题; 狄拉克激波; 二相流

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在质量和动量守恒的基础上,一维等熵二相流系统(气体流—液体流)可以被表述成如下双曲守恒律的形式[1]:

$$\begin{cases}(\alpha_1\ell_1)_t+(\alpha_1\ell_1u_1)_x=0,\\(\alpha_2\ell_2)_t+(\alpha_2\ell_2u_2)_x=0,\\(\alpha_1\ell_1u_1+\alpha_2\ell_2u_2)_t+(\alpha_1\ell_1u_1^2+\alpha_2\ell_2u_2^2+P)_x=0,\end{cases}\tag{1}$$

其中: α_j,ℓ_j 和 u_j 分别表示第 $j(j=1,2)$ 项的体积分数、密度和速度,P 代表混合压力.这里 α_1 和 α_2 满足条件 $\alpha_1+\alpha_2=1$,系统(1)就是著名的漂移流模型.流体动力闭合定律表明每项的速度符合公式 $u_2-u_1=\varphi(\alpha_1,u_1,P)$.如果假设 $u=u_1=u_2$,即 $\varphi=0$,可得到简化的二相流系统[2]:

$$\begin{cases}\rho_{1t}+(\rho_1u)_x=0,\\\rho_{2t}+(\rho_2u)_x=0,\\((\rho_1+\rho_2)u)_t+((\rho_1+\rho_2)u^2+P(\rho_1,\rho_2))_x=0,\end{cases}\tag{2}$$

这里 $\rho_1=\alpha_1\ell_1$,$\rho_2=\alpha_2\ell_2$.系统(2)中 $P(\rho_1,\rho_2)=k(\rho_1+\rho_2)^\gamma(\gamma>1)$ 时的黎曼问题和弱激波相互作用已经在 2018 年被 Minhajul 等[3] 研究.

状态方程为 Chaplygin 气体等熵二相流漂移通量方程的相关问题已经被研究.因此在本文中,令系统(2)中的状态方程为广义 Chaplygin 气体,即这里取混合压力 $P(\rho_1,\rho_2)=-\dfrac{k}{(\rho_1+\rho_2)^\alpha}(0<\alpha<1)$,则系统(2)在上述压力下可以写成如下形式:

$$\begin{cases}\rho_{1t}+(\rho_1u)_x=0,\\\rho_{2t}+(\rho_2u)_x=0,\\((\rho_1+\rho_2)u)_t+\left((\rho_1+\rho_2)u^2-\dfrac{k}{(\rho_1+\rho_2)^\alpha}\right)_x=0,\end{cases}\tag{3}$$

式中: u 代表速度,ρ_1 和 ρ_2 分别表示气体和液体的密度.很容易发现系统(3)是严格双曲的,并且其中有两个特征场是真正非线性的,而另一个特征场是线性退化的.

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为了进一步研究系统(3),我们研究该系统在如下两片常状态初始值的黎曼问题:

$$(\rho_1,\rho_2,u)(x,0)=\begin{cases}(\rho_{1-},\rho_{2-},u_-), & -\infty<x<0,\\(\rho_{1+},\rho_{2+},u_+), & 0<x<+\infty,\end{cases} \tag{4}$$

其中:$\rho_{1\pm},\rho_{2\pm},u_\pm$ 都为常数.不难发现,当 $u_- - \sqrt{\dfrac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}} \geqslant u_+ + \sqrt{\dfrac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}$ 时,系统(3)~(4)黎曼解为狄拉克激波解;否则,黎曼解是激波、稀疏波和接触间断的组合.这里狄拉克 δ-函数同时叠加在 ρ_1 和 ρ_2 上.不同形式广义 Chaplygin 气体方程的黎曼问题已经得到了广泛研究[4—8].推导得出狄拉克激波解所满足的广义 Rankine-Hugoniot 条件,并进一步得到 ρ_1 和 ρ_2 沿着狄拉克激波曲线上的速度和强度.

1 黎曼问题

在本节中,构造方程组(3)~(4)黎曼解.首先,将方程组(3)改写成如下形式:

$$\begin{bmatrix}1&0&0\\0&1&0\\u&u&\rho_1+\rho_2\end{bmatrix}\begin{bmatrix}\rho_1\\\rho_2\\u\end{bmatrix}_t+\begin{bmatrix}u&0&\rho_1\\0&u&\rho_2\\u^2+\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}&u^2+\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}&2(\rho_1+\rho_2)u\end{bmatrix}\begin{bmatrix}\rho_1\\\rho_2\\u\end{bmatrix}_x=0.$$

易得系统(3)有3个相异的特征值:

$$\lambda_1=u-\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}},\lambda_2=u,\lambda_3=u+\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}, \tag{5}$$

对应的特征向量为:

$$\boldsymbol{r}_1=\left(\rho_1,\rho_2,-\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\right)^{\mathrm T},\ \boldsymbol{r}_2=(1,-1,0)^{\mathrm T},\boldsymbol{r}_3=\left(\rho_1,\rho_2,\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\right)^{\mathrm T}. \tag{6}$$

通过计算,可以发现 $\nabla\lambda_1\boldsymbol{r}_1\neq0,\nabla\lambda_2\boldsymbol{r}_2=0,\nabla\lambda_3\boldsymbol{r}_3\neq0$,其中 $\nabla=\left(\dfrac{\partial}{\partial\rho_1},\dfrac{\partial}{\partial\rho_2},\dfrac{\partial}{\partial u}\right)$.也就是说,系统(3)是一个严格双曲型系统,并且 λ_1 和 λ_3 是真正非线性的,而 λ_2 是线性退化的.

接下来,考虑自相似变换 $\xi=\dfrac{x}{t}$,则系统(3)可以改写为:

$$\begin{cases}-\xi\rho_{1\xi}+(\rho_1u)_\xi=0,\\-\xi\rho_{2\xi}+(\rho_2u)_\xi=0,\\-\xi((\rho_1+\rho_2)u)_\xi+\left((\rho_1+\rho_2)u^2-\dfrac{k}{(\rho_1+\rho_2)^{\alpha}}\right)_\xi=0,\end{cases} \tag{7}$$

其中 $0<\alpha<1$.并且式(7)中所有的光滑解都满足:

$$\begin{bmatrix}-\xi+u&0&\rho_1\\0&-\xi+u&\rho_2\\-\xi u+u^2+\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}&-\xi u+u^2+\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}&(\rho_1+\rho_2)(-\xi+2u)\end{bmatrix}\begin{bmatrix}\rho_1\\\rho_2\\u\end{bmatrix}_\xi=0. \tag{8}$$

这里,计算得出 $\xi_1=\lambda_1=u-\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}},\xi_2=\lambda_2=u,\xi_3=\lambda_3=u+\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}$.将 ξ_1 带入(8)可以得到:

$$\begin{cases}\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\mathrm{d}\rho_1+\rho_1\mathrm{d}u=0,\\ \sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\mathrm{d}\rho_2+\rho_2\mathrm{d}u=0.\end{cases}\tag{9}$$

化简方程组(9)，一方面，有 $\dfrac{1}{\rho_1}\mathrm{d}\rho_1=\dfrac{1}{\rho_2}\mathrm{d}\rho_2$，即 $\rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+}$；另一方面，将式(9)的两个等式相加，有：

$$\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\mathrm{d}\rho_1+\sqrt{\frac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\mathrm{d}\rho_2+(\rho_1+\rho_2)\mathrm{d}u=0,$$

等价于 $-\dfrac{(\alpha+1)\sqrt{\alpha k}(\mathrm{d}\rho_1+\mathrm{d}\rho_2)}{2(\rho_1+\rho_2)^{\frac{\alpha+3}{2}}}=\dfrac{\alpha+1}{2}\mathrm{d}u$，即 $\mathrm{d}\left(\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}}\right)=\dfrac{\alpha+1}{2}\mathrm{d}u$. 化简得到：

$$u_+-\frac{2}{\alpha+1}\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}=u_--\frac{2}{\alpha+1}\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}.$$

因此，得到 (ρ_1,ρ_2,u) 平面的稀疏波曲线：

$$R_1:\begin{cases}\xi=\lambda_1=u-\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}},\\ \rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+},\\ u_--\dfrac{2}{\alpha+1}\sqrt{\dfrac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}=u_+-\dfrac{2}{\alpha+1}\sqrt{\dfrac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}.\end{cases}\tag{10}$$

将 ξ_3 带入式(8)，可以类似地得到：

$$R_3:\begin{cases}\xi=\lambda_3=u+\sqrt{\dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}},\\ \rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+},\\ u_-+\dfrac{2}{\alpha+1}\sqrt{\dfrac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}=u_++\dfrac{2}{\alpha+1}\sqrt{\dfrac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}.\end{cases}\tag{11}$$

将 ξ_2 带入式(8)，有 $\begin{cases}\rho_1\mathrm{d}u=\rho_2\mathrm{d}u=0,\\ \dfrac{\alpha k}{(\rho_1+\rho_2)^{\alpha+1}}(\mathrm{d}\rho_1+\mathrm{d}\rho_2)=0.\end{cases}$ 因此，得到 (ρ_1,ρ_2,u) 平面的接触间断曲线：

$$J_2:\xi=\lambda_2=u_+=u_-,\ \rho_{1+}+\rho_{2+}=\rho_{1-}+\rho_{2-}.\tag{12}$$

下面，考虑系统(3)的间断解，其满足下述 Rankine-Hugoniot 条件：

$$\begin{cases}\sigma[\rho_1]=[\rho_1u],\\ \sigma[\rho_2]=[\rho_2u],\\ \sigma[(\rho_1+\rho_2)u]=\left[(\rho_1+\rho_2)u^2-\dfrac{k}{(\rho_1+\rho_2)^{\alpha}}\right],\end{cases}\tag{13}$$

其中 $0<\alpha<1$，并且 ρ_1 的跳跃条件满足 $[\rho_1]=\rho_{1+}-\rho_{1-}$ 等. 化简方程组(13)，得到：

$$\begin{cases}\rho_{1+}(\sigma-u_+)=\rho_{1-}(\sigma-u_-),\\ \rho_{2+}(\sigma-u_+)=\rho_{2-}(\sigma-u_-),\\ (\rho_{1+}+\rho_{2+})(\sigma-u_+)^2-\dfrac{k}{(\rho_{1+}+\rho_{2+})^{\alpha}}=(\rho_{1-}+\rho_{2-})(\sigma-u_-)^2-\dfrac{k}{(\rho_{1-}+\rho_{2-})^{\alpha}}.\end{cases}\tag{14}$$

由式(14)的前两个等式可以看出：

$$
\begin{cases}
\rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+}(\text{或 } \sigma=u_+=u_-),\\
\sigma-u_+=\dfrac{\rho_{1-}}{\rho_{1+}}(\sigma-u_-).
\end{cases} \tag{15}
$$

一方面,式(15)的第一个等式等价于 $\rho_{2+}=\frac{\rho_{1+}\rho_{2-}}{\rho_{1-}}$,将该等式和式(15)的第二个等式代入式(14)的第三个等式,得到:

$$
\sigma=u_-\pm\sqrt{\frac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1+}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\frac{k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}.
$$

首先,将 $\sigma=u_--\sqrt{\frac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1+}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\frac{k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}$ 代入式(14)的第一个等式,经过化简,有:

$$
u_--\sqrt{\frac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1+}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\frac{k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}=u_+-\sqrt{\frac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1-}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\frac{k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}.
$$

因此,得到了如下激波曲线:

$$
S_1:\begin{cases}
\sigma=u_--\sqrt{\dfrac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1+}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\dfrac{k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}=u_+-\sqrt{\dfrac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1-}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\dfrac{k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}},\\
\rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+}.
\end{cases} \tag{16}
$$

通过类似的计算,有:

$$
S_3:\begin{cases}
\sigma=u_-+\sqrt{\dfrac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1+}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\dfrac{k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}=u_++\sqrt{\dfrac{\rho_{1-}^{\alpha}-\rho_{1+}^{\alpha}}{\rho_{1-}^{\alpha-1}(\rho_{1-}-\rho_{1+})}\cdot\dfrac{k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}},\\
\rho_{1+}\rho_{2-}=\rho_{1-}\rho_{2+}.
\end{cases} \tag{17}
$$

另一方面,当 $\sigma=u_+=u_-$ 时,将其代入式(14)的第三个等式,经过简单的计算,得到 $\rho_{1+}+\rho_{2+}=\rho_{1-}+\rho_{2-}$,亦即:

$$
J_2:\ \sigma=u_+=u_-,\ \rho_{1+}+\rho_{2+}=\rho_{1-}+\rho_{2-}. \tag{18}
$$

至此,上文构造的黎曼解是激波、稀疏波、接触间断的组合,其中 2-波始终为接触间断,而 1-波和 3-波是激波与稀疏波间的不同组合.同时,当 $u_--\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}\geqslant u_++\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}$ 时,有一个新的双曲型波产生,也就是狄拉克激波.为了构造狄拉克激波,有必要介绍狄拉克激波的定义(参阅文献[9—11]),有关狄拉克激波更一般的定义可以参阅文献[12—15].

定义 1 对于任意 $\psi(x,t)\in C_0^{\infty}(R\times R_+)$,$\Gamma$ 是由 $x=x(t)$ 确定的一条曲线,则在 Γ 上的 Dirac 质量和强度满足下述等式:

$$
\langle\beta(t)\delta_{\Gamma},\psi(x,t)\rangle=\int_0^{+\infty}\beta(t)\psi(x(t),t)\,\mathrm{d}t. \tag{19}
$$

下面,将在 $u_--\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}\geqslant u_++\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}$ 时构造黎曼问题(3)和(4)的狄拉克激波解,它将由下面的定理给出.

定理 2 当 $u_--\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}\geqslant u_++\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}$ 时,黎曼问题(3)~(4)的狄拉克激波解可以被写成下面形式:

$$
(\rho_1,\rho_2,u)(x,t)=\begin{cases}
(\rho_{1-},\rho_{2-},u_-),\ x<\sigma_\delta t,\\
(\beta_1(t)\delta(x-\sigma_\delta t),\beta_2(t)\delta(x-\sigma_\delta t),u_\delta),\ x=\sigma_\delta t,\\
(\rho_{1-},\rho_{2-},u_-),\ x>\sigma_\delta t,
\end{cases} \tag{20}
$$

式中：σ_δ 代表的狄拉克激波的传播速度，$\beta_1(t)$ 和 $\beta_2(t)$ 分别代表狄拉克激波在状态变量 ρ_1 和 ρ_2 上的强度. 为了确保式(3)的第三个方程在广义函数意义下是成立的，在直线 $x=\sigma_\delta t$ 上，令 $\frac{1}{\rho_1+\rho_2}=0$. 此外，狄拉克激波解(20)还需要满足下述广义 Rankine-Hugoniot 条件：

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}t}=\sigma_\delta(t),\\ \frac{\mathrm{d}\beta_1(t)}{\mathrm{d}t}=\sigma_\delta[\rho_1]-[\rho_1 u],\\ \frac{\mathrm{d}\beta_2(t)}{\mathrm{d}t}=\sigma_\delta[\rho_2]-[\rho_2 u],\\ \frac{\mathrm{d}((\beta_1(t)+\beta_2(t))\sigma_\delta(t))}{\mathrm{d}t}=\sigma_\delta[(\rho_1+\rho_2)u]-[(\rho_1+\rho_2)u^2-\frac{k}{(\rho_1+\rho_2)^\alpha}],\end{cases}\tag{21}$$

和过度压缩性的 δ-熵条件：

$$\lambda_{1+}\leqslant\lambda_{2+}\leqslant\lambda_{3+}\leqslant\sigma_\delta\leqslant\lambda_{1-}\leqslant\lambda_{2-}\leqslant\lambda_{3-}.\tag{22}$$

当 $\rho_{1+}+\rho_{2+}\neq\rho_{1-}+\rho_{2-}$ 时，有：

$$\begin{cases}\sigma_\delta=\frac{(\rho_{1+}+\rho_{2+})u_+-(\rho_{1-}+\rho_{2-})u_-+L}{\rho_{1+}+\rho_{2+}-\rho_{1-}-\rho_{2-}},\\ \beta_1(t)=\frac{(\rho_{1+}\rho_{2-}-\rho_{1-}\rho_{2+})(u_+-u_-)+L(\rho_{1+}-\rho_{1-})}{\rho_{1+}+\rho_{2+}-\rho_{1-}-\rho_{2-}}t,\\ \beta_2(t)=\frac{(\rho_{1-}\rho_{2+}-\rho_{1+}\rho_{2-})(u_+-u_-)+L(\rho_{2+}-\rho_{2-})}{\rho_{1+}+\rho_{2+}-\rho_{1-}-\rho_{2-}}t,\end{cases}\tag{23}$$

其中

$$L=\sqrt{(\rho_{1+}+\rho_{2+})(\rho_{1-}+\rho_{2-})(u_+-u_-)^2-k(\frac{1}{(\rho_{1+}+\rho_{2+})^\alpha}-\frac{1}{(\rho_{1-}+\rho_{2-})^\alpha})(\rho_{1+}+\rho_{2+}-\rho_{1-}-\rho_{2-})}.$$

当 $\rho_{1+}+\rho_{2+}=\rho_{1-}+\rho_{2-}$ 时，有：

$$\begin{cases}\sigma_\delta=\frac{u_++u_-}{2},\\ \beta_1(t)=\frac{1}{2}(\rho_{1+}+\rho_{1-})(u_--u_+)t,\\ \beta_2(t)=\frac{1}{2}(\rho_{2+}+\rho_{2-})(u_--u_+)t.\end{cases}\tag{24}$$

证明　对于任意测度函数 $\psi(x,t)\in C_c^\infty(R\times R_+)$，狄拉克激波解必须满足系统(3)的弱形式：

$$\begin{cases}\int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_1\psi_t+\rho_1 u\psi_x)\mathrm{d}x\mathrm{d}t=0,\\ \int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho_2\psi_t+\rho_2 u\psi_x)\mathrm{d}x\mathrm{d}t=0,\\ \int_0^{+\infty}\int_{-\infty}^{+\infty}\{(\rho_1+\rho_2)u\psi_t+((\rho_1+\rho_2)u^2-\frac{k}{(\rho_1+\rho_2)^\alpha})\psi_x\}\mathrm{d}x\mathrm{d}t=0.\end{cases}\tag{25}$$

由于式(25)的前两个等式明显成立，所以仅需要证明式(25)的第三个等式成立即可. 假设 $\sigma_\delta=\frac{\mathrm{d}x}{\mathrm{d}t}>0$，则：

$$I_3=\int_0^{+\infty}\int_{\frac{x}{\sigma_\delta}}^{+\infty}(\rho_{1-}+\rho_{2-})u_-\psi_t\mathrm{d}t\mathrm{d}x+\int_0^{+\infty}\int_{-\infty}^{\sigma_\delta t}((\rho_{1-}+\rho_{2-})u_-^2-\frac{k}{(\rho_{1-}+\rho_{2-})^\alpha})\psi_x\mathrm{d}x\mathrm{d}t+$$

$$\int_0^{+\infty}\int_0^{\frac{x}{\sigma_\delta}}(\rho_{1+}+\rho_{2+})u_+\psi_t\mathrm{d}t\mathrm{d}x+\int_0^{+\infty}\int_{-\infty}^{\sigma_\delta t}\left((\rho_{1+}+\rho_{2+})u_+^2-\frac{k}{(\rho_{1+}+\rho_{2+})^\alpha}\right)\psi_x\mathrm{d}x\mathrm{d}t+$$
$$\int_0^{+\infty}\sigma_\delta(\beta_1(t)+\beta_2(t))(\psi_t(\sigma_\delta t,t)+\sigma_\delta\psi_x(\sigma_\delta t,t))\mathrm{d}t,$$

化简得:

$$I_3=-\int_0^{+\infty}(\rho_{1-}+\rho_{2-})u_-\psi(x,\frac{x}{\sigma_\delta})\mathrm{d}x+\int_0^{+\infty}((\rho_{1-}+\rho_{2-})u_-^2-\frac{k}{(\rho_{1-}+\rho_{2-})^\alpha})\psi(\sigma_\delta t,t)\mathrm{d}t+$$
$$\int_0^{+\infty}(\rho_{1+}+\rho_{2+})u_+\psi(x,\frac{x}{\sigma_\delta})\mathrm{d}x-\int_0^{+\infty}((\rho_{1+}+\rho_{2+})u_+^2-\frac{k}{(\rho_{1+}+\rho_{2+})^\alpha})\psi(\sigma_\delta t,t)\mathrm{d}t+$$
$$\int_0^{+\infty}\sigma_\delta(\beta_1(t)+\beta_2(t))\cdot\frac{\mathrm{d}\psi(\sigma_\delta t,t)}{\mathrm{d}t}\cdot\mathrm{d}t.$$

通过变量替换,易知:

$$I_3=-\int_0^{+\infty}\sigma_\delta(\rho_{1-}+\rho_{2-})u_-\psi(\sigma_\delta t,t)\mathrm{d}t+\int_0^{+\infty}((\rho_{1-}+\rho_{2-})u_-^2-\frac{k}{(\rho_{1-}+\rho_{2-})^\alpha})\psi(\sigma_\delta t,t)\mathrm{d}t+$$
$$\int_0^{+\infty}\sigma_\delta(\rho_{1+}+\rho_{2+})u_+\psi(\sigma_\delta t,t)\mathrm{d}t-\int_0^{+\infty}((\rho_{1+}+\rho_{2+})u_+^2-\frac{k}{(\rho_{1+}+\rho_{2+})^\alpha})\psi(\sigma_\delta t,t)\mathrm{d}t-$$
$$\sigma_\delta(\beta_1'(t)+\beta_2'(t))\int_0^{+\infty}\psi(\sigma_\delta t,t)\mathrm{d}t,$$

上式等价于:

$$I_3=\{\sigma_\delta((\rho_{1+}+\rho_{2+})u_+-(\rho_{1-}+\rho_{2-})u_-)-((\rho_{1+}+\rho_{2+})u_+^2-\frac{k}{(\rho_{1+}+\rho_{2+})^\alpha})+$$
$$((\rho_{1-}+\rho_{2-})u_-^2-\frac{k}{(\rho_{1-}+\rho_{2-})^\alpha})-\sigma_\delta(\beta_1'(t)+\beta_2'(t))\}\int_0^{+\infty}\psi(\sigma_\delta t,t)\mathrm{d}t=0.$$

由此可知,当 $u_--\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}\geqslant u_++\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}$ 时,式(25)的第三个等式成立.也就是说,狄拉克激波解(20)满足分布意义下的系统(3).此外,过压缩 δ 熵条件(22)可以转换为:

$$u_++\sqrt{\frac{\alpha k}{(\rho_{1+}+\rho_{2+})^{\alpha+1}}}\leqslant\sigma_\delta\leqslant u_--\sqrt{\frac{\alpha k}{(\rho_{1-}+\rho_{2-})^{\alpha+1}}}.\qquad(26)$$

结合初始条件 $x(0)=0,\beta_1(0)=0,\beta_2(0)=0$ 和常数 σ_δ,式(23)和(24)的结果可以由式(21)和(26)直接解得.证毕.

2 结语

本文研究了一个严格双曲二相流系统的黎曼问题.除了激波、稀疏波、接触间断以外,还发现了在特定情况下会产生狄拉克激波.在后续工作中,将继续研究这一系统狄拉克激波的相互作用问题.

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Riemann Problem for the Two-phase Flow Equations with Generalized Chaplygin Gas

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Abstract: The Riemann problem for the isentropic drift-flux equations of two-phase flows under the equation of state for generalized Chaplygin gas was investigated in detail in this work, and its Riemann solution consists of either a delta shock wave or the different combination of shock wave, rarefaction wave and contact discontinuity. In the delta shock wave solution, the Dirac delta function was developed simultaneously in the two densities. Moreover, the weights of Dirac delta functions for the two densities and the propagation speed of delta shock wave were calculated explicitly by using the generalized Rankine-Hugoniot conditions.

Keywords: Riemann problem; delta shock wave; two-phase flow

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